ON THE CASIMIR ENERGY FOR A
2N-PIECE RELATIVISTIC STRING

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Abstract

The Casimir energy for the transverse oscillations of a piecewise uniform closed string is calculated. The string consists of $2N$ pieces of equal length, of alternating type I and type II material, and is taken to be relativistic in the sense that the velocity of sound always equals the velocity of light. By means of a new recursion formula we manage to calculate the Casimir energy for arbitrary integers N. Agreement with results obtained in earlier works on the string is found in all special cases. As basic regularization method we use the contour integration method. As a check, agreement is found with results obtained from the ζ function method (the Hurwitz function) in the case of low N ($N = $ 1-4). The Casimir energy is generally negative, and the more so the larger is the value of N. We illustrate the results graphically in some cases. The generalization to finite temperature theory is also given.

I. INTRODUCTION

When dealing with zero point energies in quantum field theory, it is generally desirable to deal with simple models that can be calculated explicitly, in detail. In the first place one can demonstrate the physical equivalence between different regularization schemes in this way. Secondly, as a more important point physically, such considerations can help us to understand the issue of the energy of the vacuum state in a real system, quite a compelling goal. The relativistic, piecewise uniform string model is in our opinion a model that is useful in this context. It is two-dimensional, it is easy to handle mathematically as a mechanically vibrating system, it lies open to several different regularization schemes, and finally it is easily generalizable to the case of finite temperatures. The model was introduced by Brevik and Nielsen in 1990 [1], for the most simple case of a two-piece string. Later, the model in generalized form was analysed from various points of view [2 - 5]. We shall not give a survey of these earlier developments here, but focus attention instead directly on the case where the string is divided into $2N$ pieces, of alternating type I and type II material, with N an integer. This is the case studied in [4]. The new element in our present analysis is that we shall show explicitly how the Casimir energy is found when N is *arbitrary* (in [4] we carried out the calculation in full only when $N = 2$). The key point is that we will be able to relate a $2(N + 1)$ - piece string to a $2N$-piece string by means of a recursion formula. The whole formalism becomes in turn remarkably simple. Again, we see here an example of how well the composite string model fits into the standard formalism of quantum field theory. The regularization method that we find to be the most advantageous one, is that involving contour integration (the so-called argument principle). This method was introduced in the context of Casimir calculations by von Kampen, Nijboer, and Schram [6], and was used also in [3,4,5]. One of the virtues of this method is that it is easily generalizable to the case of finite temperatures. We also show below how the Casimir energy can be found, in principle, if one uses instead the ζ-function regularization (the Hurwitz function). The equivalence is verified explicitly for the cases

when $N = 3$ and $N = 4$ (for $N = 1$ and $N = 2$ the equivalence was found earlier, respectively in [2] and in [4]).

Figure 1 shows, as an illustration, the string when $N = 6$. The total length is L. There are thus in this particular case 12 equal pieces, each of length $L/12$, of alternating type I and type II material, corresponding to tensions T_I and T_{II}. The mechanical system is for arbitrary N relativistic, in the sense that the velocity of sound everywhere equals the velocity of light:

$$v_s \; = \; (T_I/\rho_I)^{1/2} \; = \; (T_{II}/\rho_{II})^{1/2} \; = \; c \,, \tag{1}$$

ρ_I and ρ_{II} being the mass densities. We shall consider the transverse oscillations, called ψ, of the string. The boundary conditions at the functions are that ψ itself, as well as the transverse elastic force $T\partial\psi/\partial\sigma$ (σ denoting the length coordinate along the string), are continuous.

Is the any direct physical meaning of a string of this type? We are not aware of any direct application of the model, although it seems natural to suggest that such strings played a physical role in the early universe. It is quite remarkable, as a result of our analysis, that the Casimir energy is generally negative. Its absolute value increases monotonically with N. That is, if there were some sort of "phase transition" in the early universe, a string would be able to diminish its zero point energy by dividing itself into a larger number of pieces. This effect is particularly transparent from the formula for Casimir energy in the limit of $x = T_I/T_{II} \to 0$; cf. Eq. (38) below.

From a wider perspective, our composite string model is related to other string models proposed in the recent past, all of them with the main purpose of getting more physical insight into the energy spectrum and the vacuum state. For instance, Ferrer and de la Incera analysed the energy spectrum for an open and homogeneous string, with charges attached to its ends, in a magnetic background [7]. See also related papers of Odintsov, Lichtzier, and Bytsenko [8], and of Odintsov [9]. The connection between the Casimir energy phenomena and tachyon problems were studied by Nesterenko [10], and by D'Hoker, Sikivie, and Kanev [11]. Anm interesting variant of the

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composite string model is to assume a *twisted* string loop; cf. the recent paper of Bayin, Krisch, and Ozcan [12].

We put henceforth $\hbar = c = 1$. The next section deals with the contour integration method, for the case of an arbitrary integer N. The central recursion formula is given in Eq. (11); its solution is given in Eq. (15). Sec. III deals with diagonalization of the elemental matrix $\mathbf{\Lambda}$, and derives essentially the dispersion function. The basic integral expression for the Casimir energy is derived in Sec. IV. Sec. V deals with the alternative ζ function regularization technique. The generalization to finite temperature theory is given in Sec. VI. Conclusions are given in Sec. VII.

II. GENERAL FORMALISM

A. The eigenvalue problem

The string of total length L is assumed to be divided into $2N$ pieces, of alternating type I and type II material as mentioned above, and has thus $2N$ junctions which will be numbered by $j = 1, 2,..., 2N$. We introduce the symbol x for the tension ratio, and also the symbol p_N (cf. [4]):

$$x = T_I/T_{II} , \qquad p_N = \omega L/N .$$ (2)

It is convenient to introduce also another symbol α:

$$\alpha = (1 - x)/(1 + x) .$$ (3)

As shown in Ref. [4], the eigenfrequencies ω of the string are determined from the equation

$$\det\Big[\mathbf{M}_{2N}(x, p_N) - \mathbf{1}\Big] = 0 ,$$ (4)

where

$$\mathbf{M}_{2N}(x, p_N) = \prod_{j=1}^{2N} \mathbf{M}^{(j)}(x, p_N) .$$ (5)

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The component matrices can be expressed as

$$
\mathbf{M}^{(j)}(x, p_N) \;=\;
\begin{cases}
\dfrac{1+x}{2x}
\begin{pmatrix}
1 & -\alpha\, e^{-ijp_N} \\
-\alpha\, e^{ijp_N} & 1
\end{pmatrix}, & j \text{ odd} \\[2em]
\dfrac{1+x}{2}
\begin{pmatrix}
1 & \alpha\, e^{-ijp_N} \\
\alpha\, e^{ijp_N} & 1
\end{pmatrix}, & j \text{ even}
\end{cases}
\tag{6}
$$

for $j = 1, 2,..., (2N-1)$. At the last junction, for $j = 2N$, the matrix will be of a particular form (here and henceforth given an extra prime for clarity):

$$
\mathbf{M}'^{(2N)}(x, p_N) \;=\; \frac{1+x}{2}
\begin{pmatrix}
e^{-iNp_N} & \alpha\, e^{-iNp_N} \\
\alpha\, e^{iNp_N} & e^{iNp_N}
\end{pmatrix}.
\tag{7}
$$

From (5) - (7) it follows that, except from a scaling factor, the matrix $\mathbf{M}_{2N}$ will depend on x only through the variable $\alpha(x)$. It is therefore convenient to scale the matrices as $\mathbf{M}_{2N}(x, p_N) = [(1+x)^2/4x]^N \mathbf{m}_{2N}(\alpha, p_N)$. The new matrices can be calculated as

$$
\mathbf{m}_{2N}(\alpha, p_N) \;=\; \prod_{j=1}^{2N} \mathbf{m}^{(j)}(\alpha, p_N),
\tag{8}
$$

where

$$
\mathbf{m}^{(j)}(\alpha, p_N) \;=\;
\begin{pmatrix}
1 & \mp\alpha\, e^{-ijp_N} \\
\mp\alpha\, e^{ijp_N} & 1
\end{pmatrix}
\tag{9}
$$

for $j = 1, 2,..., (2N-1)$. Here the sign convention is to use $+/-$ for even/odd j. The last matrix $\mathbf{m}'^{(2N)}(\alpha, p_N)$ in (8) can be read off directly from (7).

B. Exact solution for arbitrary N

We shall now calculate the matrix $\mathbf{m}_{2N}(\alpha, p_N)$ for general N. This aim will be achieved by first establishing a recursion formula. For a string that is divided into $2(N+1)$ pieces we can, according to (8), write the scaled resultant matrix as

$$
\mathbf{m}_{2(N+1)} \;=\; \left[\mathbf{m}^{(1)} \cdot \ldots \cdot \mathbf{m}'^{(2N)} \right] \cdot \left[(\mathbf{m}'^{(2N)})^{-1} \cdot \mathbf{m}^{(2N)} \cdot \mathbf{m}^{(2N+1)} \cdot \mathbf{m}'^{(2N+2)} \right].
\tag{10}
$$

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All these matrices have p_{N+1} as their second argument. We can therefore write

$$\mathbf{m}_{2(N+1)}(\alpha, p_{N+1}) = \mathbf{m}_{2N}(\alpha, p_{N+1}) \cdot \mathbf{\Lambda}(\alpha, p_{N+1}), \tag{11}$$

where the matrix $\mathbf{\Lambda}$ is a product of four matrices, $\mathbf{\Lambda} = (\mathbf{m}'^{(2N)})^{-1} \cdot \mathbf{m}^{(2N)} \cdot \mathbf{m}^{(2N+1)} \cdot \mathbf{m}'^{(2N+2)}$, evaluated at p_{N+1}. We find that

$$\mathbf{\Lambda}(\alpha, p) = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}, \tag{12}$$

where

$$a = e^{-ip} - \alpha^2, \tag{13}$$
$$b = \alpha(e^{-ip} - 1). \tag{14}$$

It is seen that the matrix $\mathbf{\Lambda}$ does not depend on N explicitly, but only through the variable $p = p_{N+1} = \omega L/(N+1)$. This fact will enable us to give an explicit solution, since then

$$\mathbf{m}_{2N}(\alpha, p_N) = \mathbf{\Lambda}^N(\alpha, p_N). \tag{15}$$

The obvious way to continue is now to calculate the eigenvalues of $\mathbf{\Lambda}$, and express the elements of $\mathbf{M}_{2N}$ as powers of these. Before doing that, we will check the formalism for low values of N.

C. The case $N = 1$

This is the trivial case, since

$$\mathbf{M}_2 = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \frac{(1+x)^2}{4x} \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}, \tag{16}$$

which means that

$$M_{11} = \frac{(1+x)^2}{4x} \left[e^{-i\omega L} - \left(\frac{1-x}{1+x} \right)^2 \right],$$
$$M_{12} = \frac{1-x^2}{4x} \left(e^{-i\omega L} - 1 \right), \tag{17}$$

since p is here equal to ωL.

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D. The case $N = 2$

Using the general formalism, we find that

$$\mathbf{M}_4 = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \frac{(1+x)^4}{16x^2} \begin{pmatrix} a^2 + bb^* & b(a + a*) \\ b^*(a + a^*) & a^{*2} + bb^* \end{pmatrix}, (18)$$

which means that

$$\begin{aligned} M_{11} &= \frac{(1+x)^4}{16x^2} \left[(e^{-ip} - \alpha^2)^2 + 2\alpha^2(1 - \cos p) \right], \\ M_{12} &= \frac{(1+x)^2(1-x^2)}{8x^2} \left(e^{-ip} - 1 \right) (\cos p - \alpha^2), \end{aligned} \qquad (19)$$

with $p = \omega L/2$. These expressions are in agreement with Eqs. (15) in Ref. [4]. Our present notation implies a significant simplification in the expressions, as compared to those given in [4].

E. The case $N = 3$

This situation, corresponding to a six-piece string, can be analysed similarly. We obtain

$$\mathbf{M}_6 = \frac{(1+x)^6}{64x^3} \begin{pmatrix} a(a^2 + bb^*) + bb^*(a + a^*) & b[a^2 + bb^* + a^*(a + a^*)] \\ b^*[a^{*2} + bb^* + a(a + a^*)] & a^*(a^{*2} + bb^*) + bb^*(a + a^*) \end{pmatrix}.$$
$$(20)$$

III. DIAGONALIZATION

We have so far found the exact solution (15) for $\mathbf{m}_{2N}$. To calculate powers of $\mathbf{\Lambda}$, we will diagonalize this matrix. First, we see that the eigenvalues $\lambda_\pm$ of $\mathbf{\Lambda}$ are roots of the polynomial

$$P(\lambda) = \det(\mathbf{\Lambda} - \lambda\mathbf{1}) = \lambda^2 - 2(\cos p - \alpha^2)\lambda + (1 - \alpha^2)^2, \qquad (21)$$

giving

$$\lambda_\pm = \cos p - \alpha^2 \pm \left[\left(\cos p - \alpha^2 \right)^2 - (1 - \alpha^2)^2 \right]^{1/2}. \qquad (22)$$

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These eigenvalues are in general complex. Powers of the matrix $\mathbf{\Lambda}$ are

$$\mathbf{\Lambda}^N = \mathbf{K} \begin{pmatrix} \lambda_+^N & 0 \\ 0 & \lambda_-^N \end{pmatrix} \mathbf{K}^{-1} , \tag{23}$$

where $\mathbf{K}$ is a matrix whose columns consist of the eigenvectors of $\mathbf{\Lambda}$. Also note that we can calculate the determinant and the trace of $\mathbf{\Lambda}^N$ directly; from (23) we get

$$\det(\mathbf{\Lambda}^N) = \lambda_+^N \lambda_-^N , \qquad \mathrm{tr}(\mathbf{\Lambda}^N) = \lambda_+^N + \lambda_-^N . \tag{24}$$

Let us now consider the dispersion function for the string. According to (4), the dispersion function is essentially the same as $\det(\mathbf{M}_{2N} - \mathbf{1})$. Let us denote the latter function by $G_N^x(\omega)$:

$$G_N^x(\omega) = \det(\mathbf{M}_{2N} - \mathbf{1}) = \det(\mathbf{M}_{2N}) - \mathrm{tr}(\mathbf{M}_{2N}) + 1 . \tag{25}$$

We can then write $\mathbf{M}_{2N} = (1 - \alpha^2)^{-N} \mathbf{\Lambda}^N$. It follows that

$$
\begin{aligned}
G_N^x(\omega) &= (1 - \alpha^2)^{-2N} \lambda_+^N \lambda_-^N - (1 - \alpha^2)^{-N} (\lambda_+^N + \lambda_-^N) + 1 \\
&= 2 - (1 - \alpha^2)^{-N} (\lambda_+^N + \lambda_-^N) ,
\end{aligned} \tag{26}
$$

in view of the relationships $\lambda_+ \lambda_- = (1 - \alpha^2)^2$, $\lambda_+ + \lambda_- = 2(\cos p - \alpha^2)$ which follow from (21). Although the eigenvalues $\lambda_\pm$ are in general complex, as mentioned, the combinations $\lambda_+ \lambda_-$ and $(\lambda_+ + \lambda_-)$ are always real. Starting from the expression (26), we can now calculate the Casimir energy of the system.

IV. CASIMIR ENERGY. CONTOUR INTEGRATION METHOD

The Casimir energy E_N describes the effect from the nonhomogeneity of the string only, and is thus required to vanish for a uniform string. Therefore, E_N is equal to the zero- point energy for the composite string, minus the zero-point energy E_N^{I+II} for the uniform string, i.e.,

$$E_N = E_N^{I+II} - E_{uniform} . \tag{27}$$

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Because the string is assumed to be relativistic, satisfying the condition (1), it is irrelevant here whether the uniform string is made up of type I, or type II, material. The energy $E_{uniform}$ is the same in either case (cf. also the discussion on this point in Ref. [1]).

The most general and powerful way to proceed in the present case is to use the contour integration method. The starting point is the so-called argument principle, as used also in previous works [3-5]:

$$\frac{1}{2\pi i} \oint \omega \frac{d}{d\omega} \ln |g(\omega)| \, d\omega \;=\; \sum \omega_0 - \sum \omega_\infty \ . \tag{28}$$

Here $g(\omega)$ is any meromorphic function whose zeros are ω_0 and whose poles are ω_∞ inside the integration contour. We choose the contour shown in Fig. 2, and identify $g(\omega)$ with the dispersion function $g_N^x(\omega)$ for the string. The last-mentioned function is essentially the same as our function $G_N^x(\omega)$ above, defined in Eq. (25), but we will have to introduce a modifying x-dependent factor between $g_N^x(\omega)$ and $G_N^x(\omega)$ to satisfy the limiting constraint on the system (see below). Before determining this factor, let us in accordance with (27) subtract off the zero-point energy of the uniform string, corresponding to $x = 1$:

$$E_N(x) \;=\; \frac{1}{4\pi} \oint \omega \frac{d}{d\omega} \ln \left| \frac{g_N^x(\omega)}{g_N^{x=1}(\omega)} \right| \, d\omega \ . \tag{29}$$

The contribution to the integral from the semicirle in Fig. 2 is seen to vanish in the limit $R \to \infty$. The remaining integral along the imaginary frequency axis ($\xi = -i\omega$) is integrated by parts, while keeping R finite and taking advantage of the symmetry of the integrand about the origin. We get

$$E_N(x) = -\frac{R}{2\pi} \ln \left| \frac{g_N^x(iR)}{g_N^{x=1}(iR)} \right| + \frac{1}{2\pi} \int_0^R \ln \left| \frac{g_N^x(i\xi)}{g_N^{x=1}(i\xi)} \right| \, d\xi \ . \tag{30}$$

The constraint that we shall impose on the system is that the surface term vanishes in the limit of large R, i.e.,

$$\lim_{R \to \infty} \frac{g_N^x(iR)}{g_N^{x=1}(iR)} \;=\; 1. \tag{31}$$

From (22) it follows, with $p = iRL/N$, RL/N being a large quantity, that

$$\lambda_+ \simeq e^{RL/N} - 2\alpha^2 \ , \qquad \lambda_- = \mathcal{O}(e^{-RL/N}) \ . \tag{32}$$

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Then (26) yields

$$G_N^x(iR) \simeq -(1 - \alpha^2)^{-N} e^{RL} , \qquad G_N^{x=1}(iR) \simeq -e^{RL} , \qquad (33)$$

and it follows that the sought relationship between $G_N^x(\omega)$ and the dispersion function $g_N^x(\omega)$ is

$$g_N^x(\omega) = (1 - \alpha^2)^N G_N^x(\omega) = 2(1 - \alpha^2)^N - (\lambda_+^N + \lambda_-^N). \qquad (34)$$

The condition (31) is thereby satisfied.

To calculate the integral in (30) we need to know also $g_N^{x=1}(i\xi)$. In this case $\alpha = 0$, and one can easily show that $\lambda_\pm = \exp(\pm q)$, with $q = \xi L/N$. One finds that

$$g_N^{x=1}(i\xi) = -4\sinh^2(\frac{Nq}{2}) , \qquad (35)$$

and so we arrive at the following expression for the Casimir energy, for arbitrary x and an arbitrary integer N,

$$E_N(x) = \frac{N}{2\pi L} \int_0^\infty \ln \left| \frac{2(1 - \alpha^2)^N - [\lambda_+^N(iq) + \lambda_-^N(iq)]}{4\sinh^2(Nq/2)} \right| dq . \qquad (36)$$

Here the eigenvalues $\lambda_\pm$ for complex arguments are

$$\lambda_\pm(iq) = \cosh q - \alpha^2 \pm \left[(\cosh q - \alpha^2)^2 - (1 - \alpha^2)^2 \right]^{1/2} . \qquad (37)$$

Figure 3 shows how $E_N L$ (in dimensional units $E_N L/\hbar c$) varies with x for some different values of N. Since α occurs quadratically in (26) and (22), it follows that the eigenvalue spectrum of the system is invariant under the transformation $x \to 1/x$. It is therefore sufficient to show the variations in Casimir energy for the tension ratio interval $0 < x \le 1$ only. It is seen that the energy is generally negative, and the more so the larger is the integer N. A string, initially uniform corresponding to Casimir energy equal to zero, can accordingly at any time diminish its zero-point energy simply by dividing itself into a larger number of pieces of alternating type I/II material. It becomes very natural to wonder if not "phase transitions" of this sort were playing a physical role at some stage in the early universe.

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Let us consider finally the limiting case of extreme tension ratio, $x \to 0$. It turns out that this case is solvable analytically. Namely, since now $\alpha \to 1$, we see from (37) that $\lambda_- = 0$, $\lambda_+ = 4\sinh^2(q/2)$. From (36) we then get

$$E_N(0) \;=\; \frac{N}{\pi L} \int_0^\infty \ln \left| \frac{2^N \sinh^N(q/2)}{2\sinh(Nq/2)} \right| dq \;=\; -\frac{\pi}{6L}(N^2 - 1) \;. \quad (38)$$

This is quite a remarkable result. In this limiting case the Casimir energy is, apart from an additive constant, simply quadratic in N. For $N = 1$, the energy vanishes, in accordance with Eq. (22) in Ref. [1]. For $N = 2$, the energy becomes $E_2(0) = -\pi/2L$, in accordance with Eq. (27) in Ref. [4]. In Figure 4 we have plotted the Casimir energy, normalized with the energy $E_N(0)$ at $x = 0$, i.e., $-E_N(x)/E_N(0)$. The figure displays all the different values of N shown in Figure 3. Note that, within numerical accuracy, all the curves seem to collapse into one single curve. This suggests a very simple scaling of the energy with N, for arbitrary values of x.

V. ζ - FUNCTION METHOD

This powerful regularization method (for a general treatise, see Ref. [13]) can be used as an alternative to calculate the Casimir energy. We then first have to calculate the eigenvalue spectrum for ω explicitly, by solving the dispersion relation $G_N^x(\omega) = 0$ (or $g_N^x(\omega) = 0$). From (26) it follows that we have to solve the equation

$$\lambda_+^N + \lambda_-^N = 2(1 - \alpha^2)^N \quad (39)$$

with respect to ω, $\lambda_\pm$ being given by (22). It is convenient to make use of the following simple recursion formula for $S(N) \equiv \lambda_+^N + \lambda_-^N$:

$$S(N) = 2(\cos p - \alpha^2)S(N - 1) - (1 - \alpha^2)^2 S(N - 2) \;, \qquad N \geq 2. \quad (40)$$

This formula follows directly from (22). It is assumed here that the value of p is kept fixed, equal to $\omega L/N$, at all the recursion steps of (40). The initial values of $S(N)$ are $S(0) = 2$, $S(1) = \lambda_+ + \lambda_- = 2(\cos p - \alpha^2)$.

Once the eigenfrequency spectrum has become determined for some chosen value of N, the zero-point energy can be calculated as $\frac{1}{2}\sum \omega_n$, summed over all the branches. The degeneracy has to be taken into account explicitly, for each branch. The ζ function that comes into play here is the Hurwitz function, originally defined as

$$\zeta(s,a) \;=\; \sum_{n=0}^{\infty}(n+a)^{-s} \qquad\qquad (\,0 < a \le 1,\ \mathrm{Re}\ s > 1)\,. \qquad (41)$$

In practice, we need only to take into account the property

$$\zeta(-1,a) \;=\; -\tfrac{1}{2}(a^2 - a + \tfrac{1}{6}) \qquad\qquad (42)$$

of the analytically continued function (cf. Ref. [2]).

Let us check the ζ-function method in some simple cases. First, if $N = 1$ we see from (39) that $\cos p = \cos \omega L = 1$, which means $\omega = 2\pi n/L$ with $n = 1,\ 2,\ 3,...$. This is the same spectrum as for a uniform string. The Casimir energy accordingly vanishes, as it should. Next, if $N = 2$ we find from (39) and (40) that there are two branches, given by $\cos p = 1$ and $\cos p = 2\alpha^2 - 1$ respectively, with $p = \omega L/2$. This is in agreement with Eq. (21) in Ref. [4], and will not be further considered here. Let us instead put $N = 3$. Then (39) and (40) lead to the equation

$$(\cos p - 1)(2\cos p - 3\alpha^2 + 1)^2 \;=\; 0 \qquad\qquad (43)$$

for determining the allowed values of $p = \omega L/3$. There are thus two branches in this case. The first branch, corresponding to $\cos p = 1$, is degenerate. One may physically associate this degeneracy with the right-left symmetry of the uniform string. The second branch, corresponding to $\cos p = \frac{1}{2}(3\alpha^2 - 1)$, is also degenerate. Mathematically, this degeneracy occurs because of the second power of the second factor in (43). Physically, the degeneracy may be considered as a consequence of two single branches that have merged together. The solution of the second branch can be written

$$\omega \;=\; \frac{3}{L}\arccos\frac{3\alpha^2 - 1}{2} \;=\; \frac{3\pi}{L} \times \begin{cases} (\beta + 2n)\,, \\ (2 - \beta + 2n)\,, \end{cases} \qquad (44)$$

where $n = 0, 1, 2,.....$ and where β is a number lying in the interval $0 < \beta \le 2/3$. The zero-point energy of the composite string becomes

$$
\begin{aligned}
E_3^{I+II} & = 2 \times \tfrac{1}{2}\left(\frac{6\pi}{L}\right)\sum_{n=1}^{\infty} n \\
& + 2 \times \tfrac{1}{2}\left(\frac{3\pi}{L}\right)\sum_{n=0}^{\infty}(2n+\beta) + 2 \times \tfrac{1}{2}\left(\frac{3\pi}{L}\right)\sum_{n=0}^{\infty}(2n+2-\beta) , \quad (45)
\end{aligned}
$$

the prefactor 2 in each term describing the degeneracy. It is seen that it is the presence of the β-term that forces us to use the Hurwitz function instead of the Riemann function. Use of the relationship (42) now leads to

$$
E_3^{I+II} = \frac{6\pi}{L}\zeta(-1,1) + \frac{12\pi}{L}\zeta(-1,\beta/2) = -\frac{3\pi}{2L}(1-\beta)^2 , \qquad (46)
$$

where we have taken into account that $\zeta(-1, 1-a) = \zeta(-1, a)$. Subtraction of $E_{uniform} = -\pi/6L$ yields the Casimir energy for $N = 3$:

$$
E_3(x) = \frac{\pi}{6L}\left[1 - 9(1-\beta)^2\right] . \qquad (47)
$$

Explicit calculation shows that this expression gives values in agreement with those found from (36). In particular, $E_3(0) = -4\pi/3L$, in agreement with (38).

Finally, let us put $N = 4$. From (39) and (40) we then obtain

$$
(\cos p - 1)(\cos p - \alpha^2)^2(\cos p - 2\alpha^2 + 1) = 0 \qquad (48)
$$

as the equation determining $p = \omega L/4$. There are in this case three branches. The two first branches, corresponding to $\cos p = 1$ and $\cos p = \alpha^2$, are degenerate, whereas the third branch, corresponding to $\cos p = 2\alpha^2 - 1$, is not. Let us denote the two last-mentioned branches by indices 1 and 2. Thus, for the second branch we have

$$
\omega = \frac{4}{L}\arccos\alpha^2 = \frac{4\pi}{L} \times \begin{cases} \beta_1 + 2n , \\ (2 - \beta_1 + 2n) , \end{cases} \qquad (49)
$$

with $0 < \beta_1 < \tfrac{1}{2}$, whereas for the third branch

$$
\omega = \frac{4}{L}\arccos(2\alpha^2 - 1) = \frac{4\pi}{L} \times \begin{cases} \beta_2 + 2n , \\ (2 - \beta_2 + 2n) , \end{cases} \qquad (50)
$$

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with $0 < \beta_2 \leq 1$. The zero-point energy becomes

$$
\begin{aligned}
E_4^{I+II} &= 2 \times \tfrac{1}{2}\left(\frac{8\pi}{L}\right) \sum_{n=1}^{\infty} n \\
&+ 2 \times \tfrac{1}{2}\left(\frac{4\pi}{L}\right) \sum_{n=0}^{\infty}(2n + \beta_1) + 2 \times \tfrac{1}{2}\left(\frac{4\pi}{L}\right) \sum_{n=0}^{\infty}(2n + 2 - \beta_1) \\
&+ \tfrac{1}{2}\left(\frac{4\pi}{L}\right) \sum_{n=0}^{\infty}(2n + \beta_2) + \tfrac{1}{2}\left(\frac{4\pi}{L}\right) \sum_{n=0}^{\infty}(2n + 2 - \beta_2) \\
&= \frac{8\pi}{L}\zeta(-1,1) + \frac{16\pi}{L}\zeta(-1,\beta_1/2) + \frac{8\pi}{L}\zeta(-1,\beta_2/2) \\
&= -\frac{\pi}{L}\left[2(1-\beta_1)^2 + (1-\beta_2)^2 - \tfrac{1}{3}\right],
\end{aligned}
\tag{51}
$$

and so the Casimir energy becomes

$$
E_4(x) = \frac{\pi}{2L}\left[1 - 4(1-\beta_1)^2 - 2(1-\beta_2)^2\right].
\tag{52}
$$

Again, explicit evaluation leads to agreement with the integral formula (36). We see that when using the ζ-function method we have to determine the eigenfrequency spectrum explicitly, and thereafter put in the degeneracies by hand. The very useful bonus associated with our contour integration technique above, is that the degeneracies precisely correspond to the multiplicities of the zeros in the argument principle, Eq. (28), and need not be taken into account explicitly. In view of these properties, the contour integration method appears to be the simplest method in the present case.

VI. FINITE TEMPERATURE THEORY

The generalization of the $T = 0$ theory above to the case of finite temperatures is readily accomplished by starting from the integral expression (36) and replacing the integral over imaginary frequencies by a sum:

$$
\int_0^{\infty} d\xi \;\rightarrow\; 2\pi k_B T \sum_{n=0}^{\infty}{}' \qquad ,
\tag{53}
$$

13

the prime meaning that the $n = 0$ term is taken with half weight. Introducing the Matsubara frequencies $\xi_n = 2\pi n k_B T$ we then get

$$
E_N^T(x) \;=\; k_B T \sum_{n=0}^{\infty} {}' \ln \left| \frac{2(1-\alpha^2)^N - [\lambda_+^N(i\xi_n L/N) + \lambda_-^N(i\xi_n L/N)]}{4\sinh^2(\xi_n L/2)} \right| \tag{54}
$$

as the expression giving the Casimir energy, valid at any temperature T. Here, $\lambda_\pm(i\xi_n L/N)$ are given by (37), with $q \to q_n = \xi_n L/N$. It is useful to note that

$$
\lambda_+(iq_n) + \lambda_-(iq_n) = 2(\cosh q_n - \alpha^2) \;. \tag{55}
$$

There are several special cases of interest here. First, if the string is uniform ($x = 1$), we get from (54) that $E_N^T(1) = 0$. This is as we would expect, since even at finite temperatures the Casimir energy is intended to describe the influence from the inhomogeneity of the string only. Next, if $N = 1$, x arbitrary, we also get a vanishing result, $E_1^T(x) = 0$. If $N = 2$, we get for $E_2^T(x)$ the same integral expression as in Eq. (36) in Ref. [4]. For larger values of N, $N = 3, 4, ...$, we can develop the integral expressions in the same way. In particular, in the case of $x \to 0$ we get the simple formula

$$
E_N^T(0) \;=\; 2k_B T \sum_{n=0}^{\infty} {}' \ln \left| \frac{2^N \sinh^N(\xi_n L/2N)}{2\sinh(\xi_n L/2)} \right| \;. \tag{56}
$$

We shall not discuss this topic in further detail here. As shown in Ref. [3], for practical purposes the series can be evaluated fairly easily by means of a computer program.

The Casimir energy found here may be helpful also in the construction of string theory at finite temperatures (for a review, see Chapter 8 of [13]), with further possible applications in string cosmology.

VII. CONCLUSIONS

In this paper we have found the general expression for the Casimir energy for the $2N$-piece relativistic string, for an arbitrary integer N, and for arbitrary ratio between the two kinds of material. The expression for $E_N(x)$,

at temperature $T = 0$, is given in (36). The present work generalizes earlier works [1-4], and is in agreement with them in all special cases. At finite temperatures T, the corresponding Casimir energy $E_N^T(x)$ is given in (54). The key new element in our analysis is the recursion formula (11) and its explicit solution (15), which enables us to find $E_N(x)$ for arbitrary N. The use of this recursion formula greatly simplifies the calculation of the dispersion function $G_N^x(\omega)$ or $g_N^x(\omega)$; cf. Eqs. (26) and (34).

The regularization method used in the derivation of (36) was the contour integration method (argument principle), originally introduced in Casimir-type of calculations in Ref. [6]. There are three reasons why we consider this regularization method to be preferable in the present problem:

1) The method is simple, in that we do not have to solve for the eigenvalue spectrum explicitly. Moreover, we do not have to take into account the degeneracies explicitly; they are automatically accounted for, in the multiplicities of the zeros in the argument principle.

2) The generalization to arbitrary integers N is straightforward.

3) The generalization to finite temperature theory is straightforward.

As a check of the results obtained from the contour integration method, we gave in Sec.V an independent derivation based upon the ζ function method. The actual ζ function here is the Hurwitz function. When proceeding in this way, the eigenvalue spectrum has to be worked out, and the degeneracies have to be put in by hand. Explicit evaluation in the cases $N = 3$ and $N = 4$ gave results in agreement with the integral formula (36).

A remarkable physical result is that the Casimir energy is always negative, and the more so the higher is the value of N. An eclatant example of this behaviour is the expression (38), showing the Casimir energy in the case $x \to 0$. A string can always lower its zero-point energy by dividing itself into a larger number of pieces, of alternating type I / II material. Perhaps were processes of this type taking place in the early universe, as some sort of "phase transitions".

Another point worth noticing is the apparent scaling of the Casimir energy with N for arbitrary values of x, which is strongly suggested by Fig. 4.

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FIGURE CAPTIONS

FIG.1 String of length L, of alternating type I and type II material, in the case when $N = 6$.

FIG.2 Integration contour in the complex ω plane.

FIG.3 Nondimensional Casimir energy versus $x = T_I/T_{II}$ for some values of N.

FIG.4 The Casimir energy scaled with the value at $x = 0$, plotted versus $x = T_I/T_{II}$ for the *same* values of N as in Fig. 3.
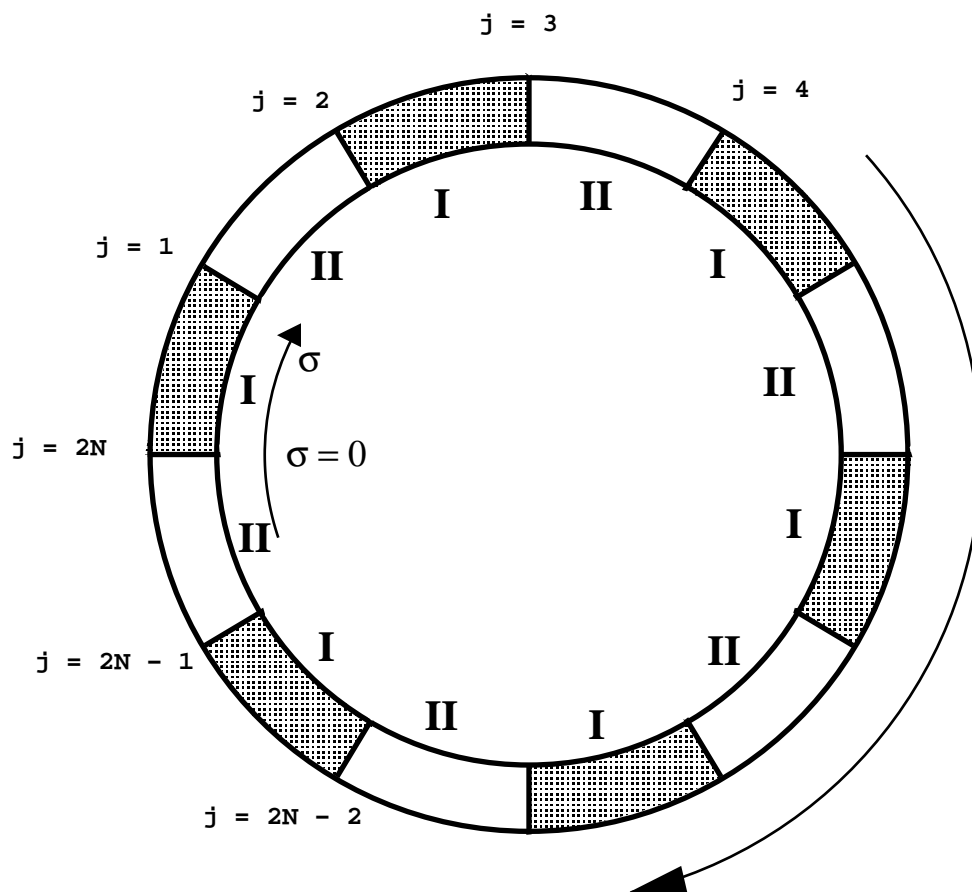
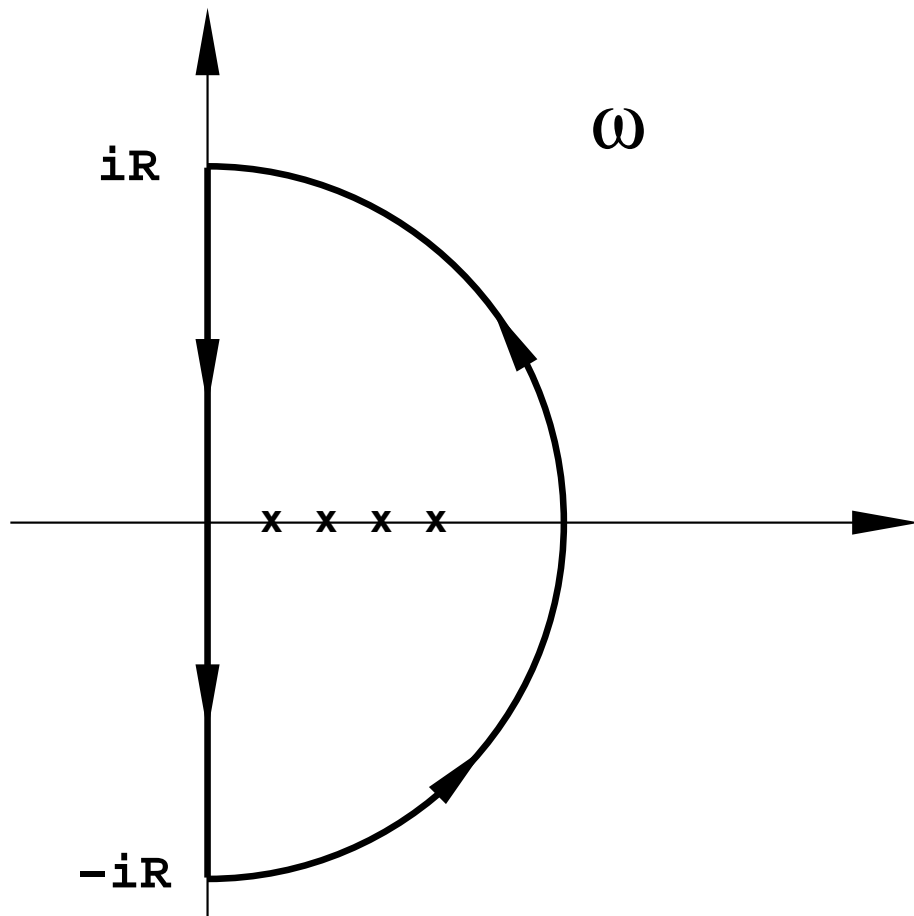Figure 1: String of length L, of alternating type I and type II material, in the case when $N = 6$.

Figure 2: Integration contour in the complex ω plane.
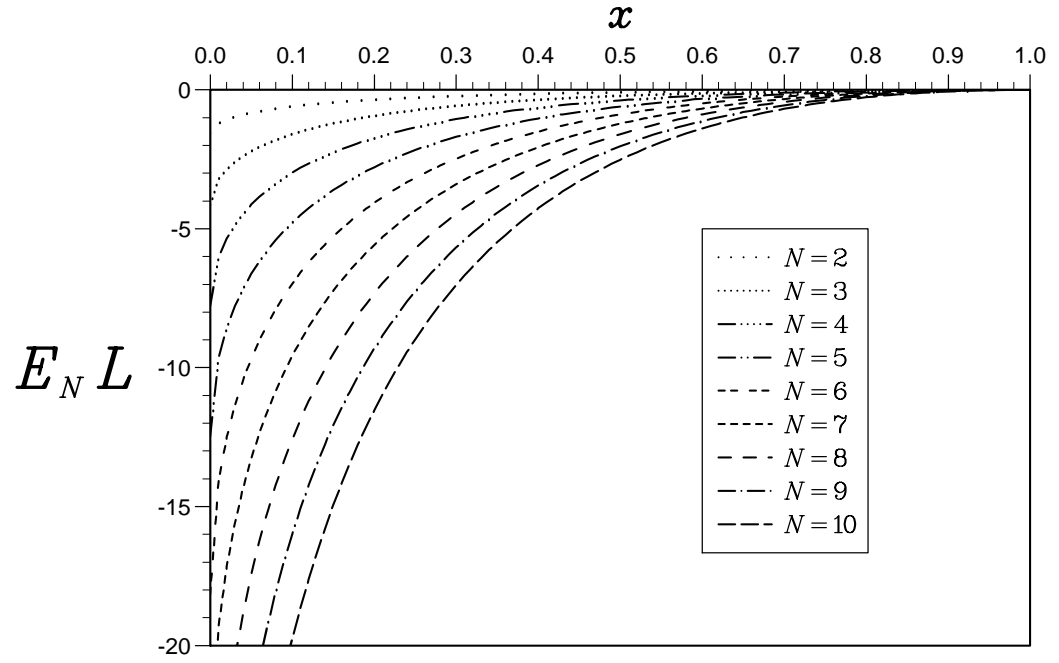
Figure 3: Nondimensional Casimir energy versus $x = T_I/T_{II}$ for some values of N.
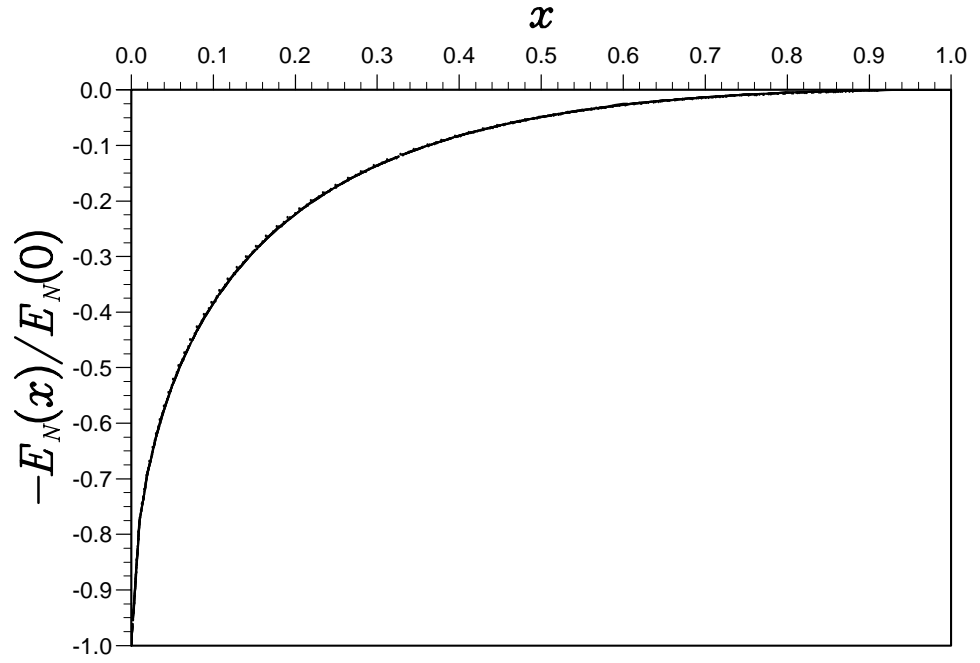
Figure 4: The Casimir energy scaled with the value at $x = 0$, plotted versus $x = T_I/T_{II}$ for the *same* values of N as in Fig. 3.

$j = 3$

$j = 2$

$j = 4$

I

II

$j = 1$

II

I

σ

I

$\sigma = 0$

II

$j = 2N$

II

I

$j = 2N - 1$

I

II

II

I

$j = 2N - 2$